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Lie Derivatives and Killing Vectors

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ABSTRACT: Followed along George Matsas Lectures Notes and Appendix C of Wald's book

Contents

1 Smooth functions and Lie Derivatives

Let N and M both be manifolds, and lets consider a diffeomorphism between them $\phi : M \to N$. And consider also an smooth function $f : N \to \mathbb{R}$. Let's discuss some key concepts;

- *Pull back* can be thought as a way of making the function f acting on a element of M , this can be achieved by composition with the diffeomorphism in the following way:
$$f \circ \phi : M \to \mathbb{R} \tag{1.1}$$

- *Push Forward*; in the same way as f, the diffeomorphism maps one point $p \in M$ to another point $\phi(p) \in N$ so naturally it also carries p's tangent vectors. By this we can define a map $\phi^* : V_p \to V_{\phi(p)}$ in which this new tangent vector acts like:
$$(\phi^* v)(f) \equiv v(f \circ \phi) \tag{1.2}$$

Also extending these concepts we might want to pull back a dual vector from $\phi(p) \in N$ to its corespondent dual vector at $p \in M$. We define the map $\phi_* : V^*_{\phi(p)} \to V^*_p$ by requiring that $\forall v^a \in V_p$:

$$(\phi_* \mu)_a v^a = \mu_a (\phi^* v)^a \tag{1.3}$$

And thus one can generalize this process to a tensors of type (k, l) both at $\phi(p)$ and p by noticing since ϕ^* is a diffeomorphism it has and inverse $(\phi^{-1})^*$ which takes vectors from $V_{\phi(p)}$ to V_p and thus by a tensor $T^{b_1 \ldots b_k}{}_{a_1 \ldots a_l}$ we define $(\phi^* T)^{b_1 \ldots b_k}{}_{a_1 \ldots a_l}$ by:

$$T^{b_1 \ldots b_k}{}_{a_1 \ldots a_l} (\phi_* \mu_1)_{b_1} \cdots ([\phi^{-1}]^* t_l)^a_l = (\phi^* T)^{b_1 \ldots b_k}{}_{a_1 \ldots a_l} (\mu_1)_{b_1} \cdots (t_l)^a_l \tag{1.4}$$

If we now have some diffeomorphism $\phi : M \to M$ this means that on our manifold we will have the same geometric properties, and statements affirmed in one frame can be translated fully to the other frame, with that we can compare tensors fields such as T and ϕ^*T, where T is a tensor field on our manifold M. One important scenario is when ϕ is a symmetry transformation for the tensor field T i.e. $\phi^*T = T$. A very special case is with the metric tensor g_{ab}, if exist some symmetry transformation for ϕ for g_{ab} such that $(\phi^*g_{ab}) = g_{ab}$, ϕ is called a *isometry*. Base on this we can also define some Gauge Freedom on general relativity.

Suppose we have two space-times described by manifold and metric (N,g) and other by (M,g'). If exist an diffeomorphism $\phi : N \to M$ and $g' = \phi^*g$ then the two space-times are said to be indistinguishable, they describe the same physics and any law or statement made in one space-time can be translated to the other, we say this is a *Gauge Freedom* in general relativity, for more detailed discussion see [3].

If we have a parameter in this diffeomorphism this would define tangent vectors v^a, since we can now compare different tensor fields by the action of ϕ, it's possible to analyse the difference between the tensor field and its diffeomorphism counterpart as the parameter goes to zero, this will be exactly the definition of the Lie derivative. The notion of a derivative of a tensor field with respect to some tangent vector v^a.

Definition 1 (Lie Derivative)**.** Let $\phi_t : M \to M$ be a diffeomorphism at one parameter and $v^a = (\frac{\mathrm{d}}{\mathrm{d}t})^a$ its tangent vector. We define the Lie derivative of a tensor field $T \in \mathcal{T}(k,l)$ by:

$$\pounds_v T = \lim_{t\to 0} \frac{\phi^*_{-t} T_{\phi(t)} - T}{t} \tag{1.5}$$

The Lie derivative have some interesting properties: proofs

- $\pounds_v T \in \mathcal{T}(k,l) \to \mathcal{T}(k,l)$
- $\pounds_v(T \otimes S) = \pounds_v T \otimes S + T \otimes \pounds_v S$
- $\pounds_v(\lambda_1 S + \lambda_2 T) = \lambda_1 \pounds_v S + \lambda_2 \pounds_v T$
- $\pounds_v\Big[C_{ij}(T)\Big] = C_{ij}(\pounds_v T)$
- $\pounds_v f = v(f)$
- $\pounds_v w = [v,w]$

Note also if the Lie derivative is zero everywhere, i.e. $\pounds_v T = 0$ if and only if ϕ_t is a symmetry transformation of T $\forall t$. Analysing the Lie derivative and also remembering the commutator using an derivative operator ∇_b we find that:

$$\pounds_v \mu_a = v^b \nabla_b \mu_a + \mu_b \nabla_a v^b \tag{1.6}$$

And for an arbitrary tensor field $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$:

$$\pounds_v T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = v^c \nabla_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} - \sum_{i=1}^{k} T^{a_1 \dots c \dots a_k}{}_{b_1 \dots b_l} \nabla_c v^{a_i} + \sum_{j=1}^{l} T^{a_1 \dots a_k}{}_{b_1 \dots c \dots b_l} \nabla_{b_j} v^c \tag{1.7}$$

2 Killing Vector Fields

In the special case where our diffeomorphism one parameter group is a isometries, i.e. $\phi_t^* g_{ab} = g_{ab}$, the vector field which generate ϕ_t , ξ^a is called a *Killing Vector Field*. Note that if ϕ_t is a group of isometries than the Lie derivative of the metric is zero, so a sufficient condition is $\pounds_\xi g_{ab} = 0$ which lead us to Killing's equation:

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{2.1}$$

Where ∇_a is the derivative operator associated with the metric.

Proposition. Let ξ^a be a Killing vector field and let γ be a geodesic with tangent u^a. Then $\xi_a u^a$ is constant along γ

Proof. We need to proof that:

$$u^b \nabla_b (\xi_a u^a) = 0 \text{ using the product rule} \tag{2.2}$$

$$u^b \nabla_b (\xi_a u^a) = u^a \left(u^b \nabla_b \xi_a \right) + \xi_a u^b \nabla_b u^a \tag{2.3}$$

Right hand side is zero because its just the geodesic equation. Left hand side generates the contraction between two tensors. One tensor is totally symmetric $u^a u^b = u^b u^a$ and the other is totally antysymmetric by Killing's Equation $\nabla_b \xi_a = -\nabla_a \xi_b$. If we remember the contraction between a symmetric and antysymmetric tensor is always zero because we write:

$$S^{(ab)} A_{[ab]} = -S^{(ab)} A_{[ba]} = -S^{(ba)} A_{[ba]} \tag{2.4}$$

If we rename our dummy indices we ended up getting

$$S^{(ab)} A_{[ab]} = -S^{(ab)} A_{[ab]} \tag{2.5}$$

The only number which satisfies that is zero. And thus:

$$u^b \nabla_b (\xi_a u^a) = 0 \tag{2.6}$$

Killing Vector fields define orbits in which geometric properties of the manifold M do not change. Also, we can use it do describe free falling particles which describe a timelike geodesic and light rays , described by null geodesics (remember these definitions work around the tangent vectors norm, i.e. $|u| = g_{ab} u^a u^b$). So we can say that our symmetry ϕ_t give rise to a conserved quantity for particles and light rays. □

Using the Riemann Tensor definition we have:

$$\nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c = R_{abc}{}^d \xi_d \tag{2.7}$$

However if we use Killing's Equation;

$$\nabla_a \nabla_b \xi_c + \nabla_b \nabla_c \xi_a = R_{abc}{}^d \xi_d \tag{2.8}$$

Now doing this cyclic permutation and summing for $(abc) + (bca) - (cab)$ we arrive at the following expression:

$$\begin{aligned} 2\nabla_b \nabla_c \xi_a &= \left(R_{abc}{}^d + R_{bca}{}^d - R_{cab}{}^d \right) \xi_d \\ &= -2R_{cab}{}^d \xi_d \end{aligned} \tag{2.9}$$

So for any killing field:

$$\nabla_a \nabla_b \xi_c = -R_{bca}{}^d \xi_d \tag{2.10}$$

This means that killing field are completely determined by their values ξ^a and $\nabla_a \xi_b$ at any point $p \in M$, if we want to know their values at another point $q \in M$ we solve the system of ordinary differential equations:

$$\begin{cases} v^a \nabla_a \xi_b = v^a L_{ab} \\ v^a \nabla_a L_{bc} = -R_{bca}{}^d \xi_d v^a \end{cases} \tag{2.11}$$

Where $L_{ab} \equiv \nabla_a \xi_b$ and v^b is the tangent of the curve connecting points p and q on the manifold M. We also have the following corollaries:

- If Killing fields and its derivative vanish at some point, so they vanish everywhere
- If $\dim M = n$ then, we have at most $\dfrac{n(n+1)}{2}$ LI killing fields and parameters group of isometries.

Another useful quantity which we can define is some current in terms of killing vector and energy momentum tensor:

$$J^\mu = -T^{\mu\nu} \xi_\nu \tag{2.12}$$

But energy momentum tensor obeys:

$$T_{\mu\nu} = T_{\nu\mu} \text{ and } \nabla_\mu T^\mu_\nu = 0 \tag{2.13}$$

With these we have for the current:

$$-\nabla_\mu J^\mu = \left(\nabla_\mu T^{\mu\nu} \right) \xi_\nu + T^{\mu\nu} \nabla_\mu \xi_\nu = 0 \tag{2.14}$$

By Stokes Theorem then:

$$\int_r \nabla_\mu J^\nu = 0 \tag{2.15}$$

Which we can decompose in:

$$\int_{\partial r} n^\mu J_\mu = \int_{\Sigma_1} n^\mu(1) J_\mu + \int_{\Sigma_2} n^\mu(2) J_\mu = 0 \tag{2.16}$$

$$\int_{\Sigma_1} n^\mu(1) J_\mu = \int_{\Sigma_2} -n^\mu(2) J_\mu \tag{2.17}$$

Which implies that $n^\mu J_\mu$ is a quantity which is conserved , where n^μ is a unitary time-like vector. One good example is to search for the killing vectors of the Minkowisky space, to search for symmetries is to solve $[\pounds_\xi g]_{\mu\nu} = 0$ for the minkowisky metric. The killing equation then writes

$$\partial_\mu \xi_\eta + \partial_\nu \xi_\mu = 0 \tag{2.18}$$

If we derive and using killing property of permutation

$$\begin{cases} \partial_\mu \partial_\lambda \xi_\nu + \partial_\lambda \partial_\nu \xi_\mu = 0 \\ -\partial_\nu \partial_\mu \xi_\lambda - \partial_\mu \partial_\lambda \xi_\nu = 0 \\ \partial_\lambda \partial_\nu \xi_\mu + \partial_\nu \partial_\mu \xi_\lambda = 0 \end{cases} \tag{2.19}$$

Summing over the 3 equations we arrive

$$\partial_\lambda \partial_\nu \xi_\mu = 0 \tag{2.20}$$

Which implies the killing vector field is something of the form:

$$\xi^\mu = a_\nu{}^\mu x^\nu + b^\mu \tag{2.21}$$

Where $a_\nu{}^\mu$ and b^μ are constants. Using the metric we find:

$$\xi_\lambda = \eta_{\lambda\mu} \xi^\mu = \eta_{\lambda\mu} a_\nu{}^\mu x^\mu + \eta_{\lambda\mu} b^\mu = a_{\lambda\nu} x^\nu + b_\lambda \tag{2.22}$$

Plugging into killing's equation once again we find

$$\partial_\mu \left(a_{\nu\lambda} x^\lambda + b_\nu \right) + \partial_\nu \left(a_{\mu\sigma} x^\sigma + b_\mu \right) \tag{2.23}$$

$$a_{\mu\nu} = -a_{\nu\mu} \tag{2.24}$$

This take us to 6 independent parameters on tensor a and 4 parameters on vector b which leed us to 10 total symmetries in Minkowyski space which are:

- 3 Boost Symmetry (x, y, z)
- 3 Rotation Symmetry (x, y, z)
- 4 Translation symmetries (t, x, y, z)

We can plot the vector field for Boost and Rotation by considering individual and independent parameters on $a_\nu{}^\mu$:

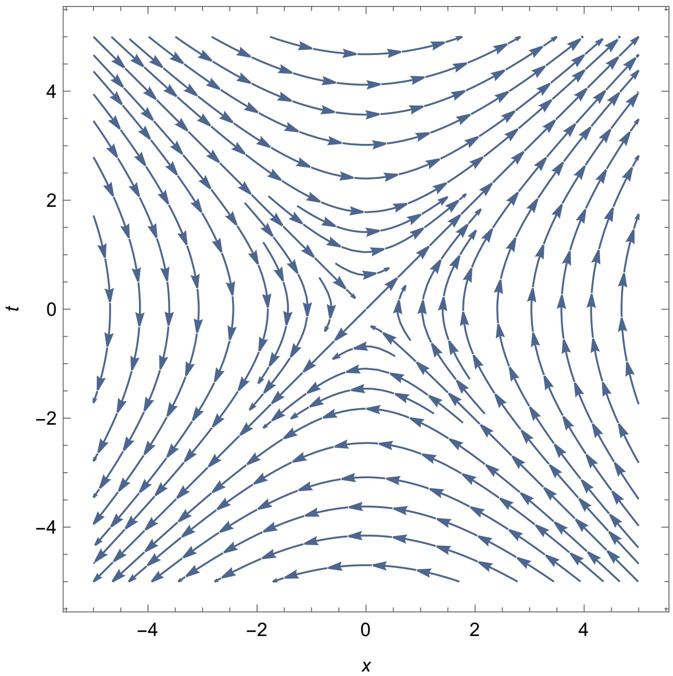

Figure 1. Lorentz Boost on the x direction vector field

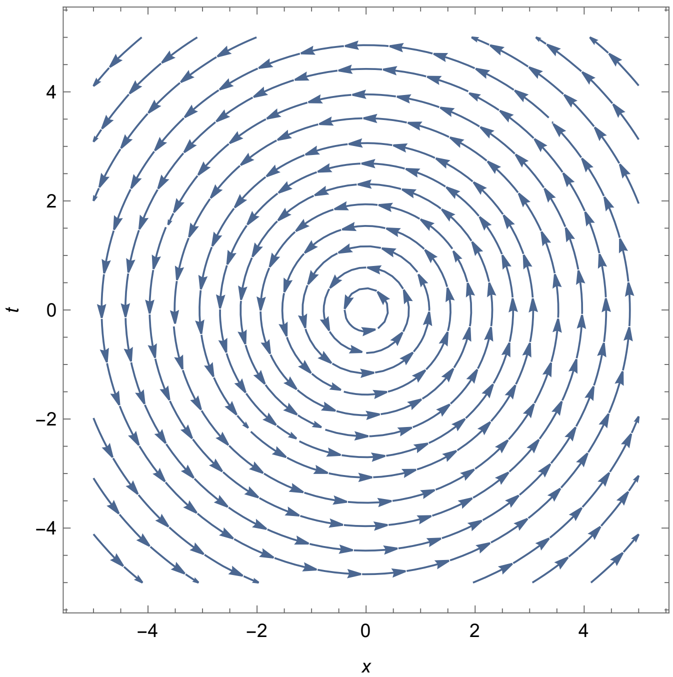

Figure 2. Rotation symmetry field for plane tx

References

[1] Matsas, George, *Notas de Aula Relatividade Geral*

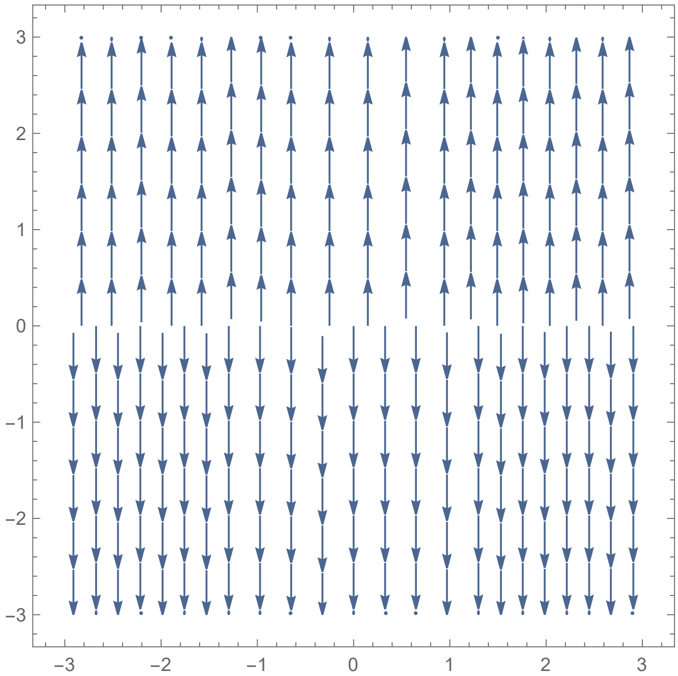

Figure 3. Temporal translation symmetry field

[2] Wald, Robert *General Relativity*,

[3] Rodrigues, João Lucas , *Einstein's Equation*