Heisenberg Uncertainty Relations:

expectation value of an operator: $\langle \Omega \rangle = \langle \psi | \Omega | \psi \rangle$

with uncertainty given by: $(\Delta \Omega)^2 = \langle \psi | (\Omega - \langle \Omega \rangle)^2 | \psi \rangle$

Derivation of Uncertainty Relations: we begin with two hermitian operators Ω and Λ with a comutator of some type: $[\Omega, \Lambda] = i\Gamma$

The product of uncertainty can be written as: (for a state $|\psi\rangle$)

$$(\Delta\Omega)^2 (\Delta\Lambda)^2 = \langle \psi | (\Omega - \langle \Omega \rangle)^2 | \psi \rangle \langle \psi | (\Lambda - \langle \Lambda \rangle)^2 | \psi \rangle$$

Changing variables to: $\underline{\tilde{\Omega} = \Omega - \langle \Omega \rangle}$ and $\underline{\tilde{\Lambda} = \Lambda - \langle \Lambda \rangle}$

Notice that $[\tilde{\Omega}, \tilde{\Lambda}] = \Omega\Lambda - \Omega\langle\Lambda\rangle - \langle\Omega\rangle\Lambda + \langle\Omega\rangle\langle\Lambda\rangle - (\Lambda\Omega - \Lambda\langle\Omega\rangle - \langle\Lambda\rangle\Omega + \langle\Lambda\rangle\langle\Omega\rangle)$

$\overset{!}{=} \Omega\Lambda - \Lambda\Omega = \underline{[\Omega, \Lambda] = i\Gamma}$, since $\langle \Omega \rangle$ and $\langle \Lambda \rangle$ are numbers and so they comute.

So:

$$(\Delta\Omega)^2 (\Delta\Lambda)^2 = \langle \psi | \tilde{\Omega}^2 | \psi \rangle \langle \psi | \tilde{\Lambda}^2 | \psi \rangle = \langle \psi | \tilde{\Omega}^\dagger \tilde{\Omega} | \psi \rangle \langle \psi | \tilde{\Lambda}^\dagger \tilde{\Lambda} | \psi \rangle$$

$$\overset{!}{=} \langle \tilde{\Omega}\psi | \tilde{\Omega}\psi \rangle \langle \tilde{\Lambda}\psi | \tilde{\Lambda}\psi \rangle$$

If we use Schwarz inequality: $|V_1|^2 |V_2|^2 \geq |\langle V_1 | V_2 \rangle|^2$ to $|\tilde{\Omega}\psi\rangle$ and $|\hat{\Lambda}\psi\rangle$:

$$\Rightarrow \underbrace{\langle \tilde{\Omega}\psi | \tilde{\Omega}\psi \rangle \langle \tilde{\Lambda}\psi | \hat{\Lambda}\psi \rangle}_{= (\Delta\Omega)^2 (\Delta\Lambda)^2} \geq |\langle \tilde{\Omega}\psi | \tilde{\Lambda}\psi \rangle|^2$$

$$\Rightarrow (\Delta\Omega)^2 (\Delta\Lambda)^2 \geq |\langle \tilde{\Omega}\psi | \tilde{\Lambda}\psi \rangle|^2$$

$$(\Delta\Omega)^2 (\Delta\Lambda)^2 \geq |\langle \psi | \tilde{\Omega}\tilde{\Lambda} | \psi \rangle|^2$$

However we can rewrite $\tilde{\Omega}\tilde{\Lambda}$ using the comutator and anticommutator:

$$\tilde{\Omega}\tilde{\Lambda} = \frac{\tilde{\Omega}\tilde{\Lambda}+\tilde{\Lambda}\tilde{\Omega}}{2} + \frac{\tilde{\Omega}\tilde{\Lambda}-\tilde{\Lambda}\tilde{\Omega}}{2} = \underbrace{\frac{1}{2}[\tilde{\Omega},\tilde{\Lambda}]}_{\text{Commutator}} + \underbrace{\frac{1}{2}[\tilde{\Omega},\tilde{\Lambda}]_+}_{\text{anticommutator}}$$

Then:

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \left|\langle\psi|\frac{1}{2}[\tilde{\Omega},\tilde{\Lambda}]+\frac{1}{2}[\tilde{\Omega},\tilde{\Lambda}]_+|\psi\rangle\right|^2$$

$$\geq \left|\langle\psi|\frac{1}{2}[\tilde{\Omega},\tilde{\Lambda}]|\psi\rangle+\langle\psi|\frac{1}{2}[\tilde{\Omega},\tilde{\Lambda}]_+|\psi\rangle\right|^2$$

$$\geq \frac{1}{4}\left|\langle\psi|[\tilde{\Omega},\tilde{\Lambda}]|\psi\rangle+\langle\psi|[\tilde{\Omega},\tilde{\Lambda}]_+|\psi\rangle\right|^2$$

remembering that $[\tilde{\Omega},\tilde{\Lambda}]=[\Omega,\Lambda]=i\Gamma$

$$\geq \frac{1}{4}\left|i\langle\psi|\Gamma|\psi\rangle+\langle\psi|[\tilde{\Omega},\tilde{\Lambda}]_+|\psi\rangle\right|^2$$

Since we have something which is $\langle\dots\rangle$ these are all numbers; which means our last expression resulted in something of the sort: $|a+ib|^2 = a^2+b^2$ then:

$$\boxed{(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{1}{4}\langle\psi|\Gamma|\psi\rangle^2 + \frac{1}{4}\langle\psi|[\tilde{\Omega},\tilde{\Lambda}]_+|\psi\rangle^2}$$

General Uncertainty Relation

This is the uncertainty relation for any given two hermitian operators; it's the most general case. We could look now on some important specific category which is the canonically conjugate operators where $\Gamma=\hbar$

In this case we would have the following:

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \frac{\hbar^2}{4}\underbrace{\langle\psi|\psi\rangle}_{\substack{=1 \\ \text{for normalization}}}^2 + \frac{1}{4}\langle\psi|[\tilde{\Omega},\tilde{\Lambda}]_+|\psi\rangle^2$$

$$\geq \frac{\hbar^2}{4} + \frac{1}{4}\underbrace{\langle\psi|[\tilde{\Omega},\tilde{\Lambda}]_+|\psi\rangle^2}$$

however notice how this term is always posite and defined; this means that this term is always <u>adding</u> something to this side of the inequality. So we can conclude that the next inequality will always be satisfited:

$$(\Delta\Omega)^2(\Delta\Lambda)^2 \geq \hbar^2/4 \quad \text{or} \quad \boxed{\Delta\Omega\,\Delta\Lambda \geq \hbar/2}$$

which is the ~~uncertainty~~ Uncertainty for cannocaly Conjugatted operators (i.e $[\Omega,\Lambda] = i\hbar$)

The equality (minimum uncertainty) happens only if:

(i) $\tilde{\Omega}|\psi\rangle = c\tilde{\Lambda}|\psi\rangle$

(ii) $\langle\psi|[\tilde{\Omega},\tilde{\Lambda}]_+|\psi\rangle = 0$

For $\hat{X}$ and $\hat{P}$ for exemple (i) translates into $(\hat{P} - \langle\hat{P}\rangle)|\psi\rangle = c(\hat{X} - \langle x\rangle)|\psi\rangle$

and (ii) into $\langle\psi|(\hat{P} - \langle\hat{P}\rangle)(\hat{X} - \langle\hat{x}\rangle) + (\hat{x} - \langle\hat{x}\rangle)(\hat{P} - \langle\hat{P}\rangle)|\psi\rangle = 0$

projecting (i) into $|x\rangle$ basis:

$$\left(-i\hbar\frac{d}{dx} - \langle\hat{p}\rangle\right)\psi(x) = c(x - \langle\hat{x}\rangle)\psi(x) \longrightarrow \frac{d\psi(x)}{\psi(x)} = \frac{i}{\hbar}\left[\langle p\rangle + c(x - \langle x\rangle)\right]dx$$

Independently of $\langle\hat{x}\rangle$ we can shift the origin to $x = \langle\hat{x}\rangle$ making $\langle\hat{x}\rangle = 0$ in this new framework.

Our solution is then of the type:

$$\psi(x) = \psi(0)\, e^{i\langle p\rangle x/\hbar}\, e^{icx^2/2\hbar}$$

Looking at (ii) ~~m~~ now but with a framework such that $\langle\hat{x}\rangle = 0$ we have:

$$\langle\psi|(\hat{P}-\langle\hat{P}\rangle)\hat{x} + \hat{x}(\hat{P}-\langle\hat{P}\rangle)|\psi\rangle = 0$$

However we must have $(\hat{P}-\langle\hat{P}\rangle)|\psi\rangle = c(\hat{x}-\langle\hat{x}\rangle)|\psi\rangle$

$$= c\hat{x}|\psi\rangle$$

where it's adjoint takes the form:

$$\langle\psi|(\hat{P}-\langle\hat{P}\rangle) = \langle\psi|\hat{x}c^*$$ so we have:

$$\langle\psi|c^*\hat{x}^2 + c\hat{x}^2|\psi\rangle = 0 \Rightarrow (c+c^*)\langle\psi|\hat{x}^2|\psi\rangle = 0$$

$\Rightarrow c = i|c|$ (pure imaginary) then:

$$\psi(x) = \psi(0)\, e^{i\langle\hat{P}\rangle x/\hbar}\, e^{-|c|x^2/2\hbar} \;;\quad \Delta^2 = \hbar/|c|$$

$$\psi(x) = \psi(0)\, e^{i\langle\hat{P}\rangle x/\hbar}\, e^{-x^2/2\Delta^2}$$ if the framework transformation was not done.

$$\psi(x) = \psi(\langle x\rangle)\, e^{i\langle\hat{P}\rangle(x-\langle x\rangle)/\hbar}\, e^{-(x-\langle\hat{x}\rangle)^2/2\Delta^2}$$

Gaussian function. If we have that our system has minimum uncertainty ~~in~~ in $\hat{x},\hat{P}$ our wave function will assume a gaussian form

(Notice that the imaginary part vanishes on the probability distribution, resulting in a arbitrary Gaussian function)

i.e.g $|\psi(x)|^2 = |\psi(\langle x\rangle)|^2 e^{-(x-\langle x\rangle)^2/\Delta^2}$

Identical Particles:

↳ particles who shares the exact same defining properties

If we have a system of two identical particles and we do some measurement of position and are returned the values <u>a</u> and <u>b</u>; we know that our state colapsed into $|x_1 x_2\rangle$. However how do we know which particle is at $x=a$ and which is at $x=b$ if they are identical? In other words, our state <u>colapsed to</u> $|ab\rangle$ or $|ba\rangle$?

In classical mechanics if we trace out ~~the~~ the path, we can 'distinguished' two identical particles. However this is not possible in quantum mechanics. The answer is neither. Our state will be a ~~s~~ vectorial superposition of $|ab\rangle$ and $|ba\rangle$ states.

$$|\psi(a,b)\rangle = \beta|ab\rangle + \gamma|ba\rangle$$

However, we need to require that any multiplyer of our state still represents our state because they need to be physically equivalent.

$$|\psi(a,b)\rangle = \alpha|\psi(b,a)\rangle \quad \text{then we will have:}$$

$$\beta|ab\rangle + \gamma|ba\rangle = \alpha\beta|ba\rangle + \alpha\gamma|ab\rangle \Rightarrow \beta = \alpha\beta \; ; \; \gamma = \alpha\gamma$$

$$\Rightarrow \alpha = \pm 1$$

So we have that the allowed ~~set~~ state vectors are:

$$|ab,S\rangle = |ab\rangle + |ba\rangle \; ; \; \alpha = 1, \text{ Symmetric}$$

$$|ab,A\rangle = |ab\rangle - |ba\rangle \; ; \; \alpha = -1, \text{ Antisymmetric}$$

In general if we measure Ω (non-degenerate) of identical particles; our state will colapse to $|\omega_1\omega_2,S\rangle$ or $|\omega_1\omega_2,A\rangle$ and this will depend soly on the type of particle we were working with.

Bosons and Fermions:

pion, photon, graviton $\rightarrow$ Bosons $\rightarrow$ Symmetric

~~the~~ electron, proton, neutron $\rightarrow$ Fermions $\rightarrow$ Antisymmetric

Considering ~~ma~~ now a two-Fermion state which is always antisymmetric:

$$|\omega_1\omega_2, A\rangle = |\omega_1\omega_2\rangle - |\omega_2\omega_1\rangle$$

If $\omega_1 = \omega_2 = \omega \Rightarrow \underline{|\omega\omega A\rangle = |\omega\omega\rangle - |\omega\omega\rangle = 0}$

$\hookrightarrow$ Pauli Exclusion Principle: two identical fermions can't be in the same quantum state

One big exemple of Pauli Exclusion Principle is ~~ina~~ a two fermionic system of spin 1/2 characterized by ω (orbital) and s (spin label) we would have the following state:

$$|\omega_1 s_1, \omega_2 s_2, A\rangle = |\omega_1 s_1, \omega_2 s_2\rangle - |\omega_2 s_2, \omega_1 s_1\rangle$$

and this vector state vanish only if $\underline{\omega_1 = \omega_2 \text{ and } s_1 = s_2}$ which is the exclusion Principle. But notice here that two eletrons cannot share the same energy level and spin orientation; thus it's possible to have two electrons on the same orbital states but with different spin orientation.

The normalization for bosonic systems are prety straight foward:

$$|\omega_1\omega_2, S\rangle = \frac{1}{\sqrt{2}}\left(|\omega_1\omega_2\rangle + |\omega_2\omega_1\rangle\right)$$

$$P_S(\omega_1, \omega_2) = |\langle\omega_1\omega_2, S|\psi_S\rangle|^2$$

Absolute probability of finding the particles in state $|\omega_1\omega_2, S\rangle$ when Ω is measured in a state $|\psi_S\rangle$

The normalization of the state can be ~~the~~ viewd as:

$$\langle \psi_S | \psi_S \rangle = 1 = \sum_{dist} | \langle \omega_1 \omega_2, S | \psi_S \rangle |^2 = \sum_{dist} P_S(\omega_1, \omega_2)$$

↑ sum over distinct states

For continuos variables we have:

$$| x_1 x_2, S \rangle = \frac{1}{\sqrt{2}} \left(| x_1 x_2 \rangle + | x_2 x_1 \rangle \right)$$

$$1 = \iint P_S(x_1, x_2) \frac{dx_1 dx_2}{2} = \iint | \langle x_1 x_2, S | \psi_S \rangle |^2 \frac{dx_1 dx_2}{2}$$

We need this ½ factor due to double conting of states on the integral

For convenioos we define de wave function as:

$$\psi_S(x_1, x_2) = \frac{1}{\sqrt{2}} \langle x_1 x_2, S | \psi_S \rangle$$

So:

$$1 = \iint | \psi_S(x_1, x_2) |^2 dx_1 dx_2$$

$$\rightarrow \quad P_S(x_1, x_2) = 2 | \psi_S(x_1, x_2) |^2$$

Exemple: Two non-interacting Bosons in a box size L. Energy measure enconted n=3 and n=4 levels. The state after the measurament is

$$| \psi_S \rangle = \frac{|3,4\rangle + |4,3\rangle}{\sqrt{2}} \quad \text{with wave function}$$

$$\psi_S(x_1, x_2) = \frac{1}{\sqrt{2}} \langle x_1 x_2, S | \psi_S \rangle = \frac{1}{2} \left(\langle x_1 x_2 | + \langle x_2 x_1 | \right) \left(\frac{|3,4\rangle + |4,3\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{2\sqrt{2}} \Big(\underbrace{\langle x_1 x_2 | 3,4 \rangle}_{\psi_3(x_1)\psi_4(x_2)} + \underbrace{\langle x_1 x_2 | 4,3 \rangle}_{\psi_4(x_1)\psi_3(x_2)} + \underbrace{\langle x_2 x_1 | 3,4 \rangle}_{\psi_3(x_2)\psi_4(x_1)} + \underbrace{\langle x_2 x_1 | 4,3 \rangle}_{\psi_4(x_2)\psi_3(x_1)} \Big)$$

$$= \frac{1}{\sqrt{2}} \left(\psi_3(x_1)\psi_4(x_2) + \psi_4(x_1)\psi_3(x_2) \right) \quad \text{where } \psi_n = \frac{\sqrt{2}}{\sqrt{L}} \sin\left(\frac{n\pi x}{L} \right)$$

For Fermions we have the following:

$$|\omega_1 \omega_2, A\rangle = \frac{1}{\sqrt{2}}\left(|\omega_1\omega_2\rangle - |\omega_2\omega_1\rangle\right)$$

with wave-function:

$$\psi_A(x_1,x_2) = \frac{1}{\sqrt{2}}\langle x_1 x_2 A|\psi_A\rangle$$

$$P_A(x_1,x_2) = 2|\psi_A(x_1,x_2)|^2 \text{ with normalization}$$

$$1 = \iint P_A(x_1,x_2)\frac{dx_1dx_2}{2} = \iint |\psi_A(x_1,x_2)|^2 dx_1dx_2$$

In the same exemple but with fermions:

$$|\psi_A\rangle = \frac{|4,3\rangle - |3,4\rangle}{\sqrt{2}} \quad \text{the wave function is}$$

$$\psi_A(x_1,x_2) = \frac{1}{\sqrt{2}}\langle x_1 x_2, A|\psi_A\rangle = \frac{1}{\sqrt{2}}\left(\psi_3(x_1)\psi_4(x_2) - \psi_4(x_1)\psi_3(x_2)\right)$$

$$\psi_A(x_1,x_2) = \frac{1}{\sqrt{2}}\begin{vmatrix} \psi_3(x_1) & \psi_4(x_1) \\ \psi_3(x_2) & \psi_4(x_2) \end{vmatrix}$$

Many Particles:

Entangled states → states with two or more particles 'correlated'

Tensor Product of two vector spaces:

Let H_1^N, H_2^M be two hilbert vector spaces of dimensions N and M. ~~Both~~ Each represent some quantum particle. How can we construct a space such ~~that~~ that the system of ~~two~~ some quantum particle represents simultanisly these two particles? → By the tensor product.

Suppose system 1 has a state vector $|\varphi\rangle \in H_1^N$ and system 2 has a state vector $|\psi\rangle \in H_2^M$. Our global system which represents both systems will be the pair $\underline{\{|\varphi\rangle, |\psi\rangle\}} \in H_1^N \otimes H_2^M$ (vector space of dimension NM)

Let's have now some orthonormal basis of each H_1^N and H_2^M:

$|\varphi\rangle = \sum_{n=1}^{N} c_n |n\rangle$; $|\psi\rangle = \sum_{m=1}^{M} d_m |m\rangle$; Our tensor product will be a space of the pair $\{|n\rangle, |m\rangle\}$ which we will denote by: $\underline{|n \otimes m\rangle \,;\, |n\rangle \otimes |m\rangle \,;\, |nm\rangle}$

And will also be orthonormal: $\langle n' \otimes m' | n \otimes m \rangle = \delta_{nn'} \delta_{m'm}$

So the new state vector which represents both system will be:

$$|\varphi \otimes \psi\rangle = |\varphi\rangle \otimes |\psi\rangle = \sum_{n,m} c_n d_m |n \otimes m\rangle$$

note: linearity:

$$|\varphi \otimes (\psi_1 + \lambda \psi_2)\rangle = |\varphi \otimes \psi_1\rangle + \lambda |\varphi \otimes \psi_2\rangle$$

$$|(\varphi_1 + \lambda \varphi_2) \otimes \psi\rangle = |\varphi_1 \otimes \psi\rangle + \lambda |\varphi_2 \otimes \psi\rangle$$

note: Not every element of $H_1^N \otimes H_2^M$ can be written as a direct or tensor product.

i.eg: $\underset{H_1^N \otimes H_2^M}{\underset{\cap}{|\psi\rangle}} = |x_1'\rangle \otimes |x_2'\rangle + |x_1''\rangle \otimes |x_2''\rangle \neq \underset{H_1^N \otimes H_2^M}{\underset{\cap}{|\psi\rangle}} = |\psi_1\rangle \otimes |\psi_2\rangle$

this is not a direct product — This is a direct product

States in which cannot be written as direct products are called Entangled states and also represents interactions between particles/systems.

But what about operators acting on only one space? Such $\hat{X}_1 \in H_1^N$ or $\hat{P}_2 \in H_2^M$ what would be their representation on the tensor product $H_1^N \otimes H_2^M$? How would they act on their elements? By the following: I_1, I_2 are each identity operator on their respective space; so:

$$\underbrace{\hat{X}_1^{H_1^N \otimes H_2^M}}_{\substack{\text{Representation of } \hat{X}_1 \in H_1^N \\ \text{on } H_1^N \otimes H_2^M}} \Rightarrow \quad (\hat{X}_1 \otimes I_2) | n \otimes m \rangle = \hat{X}_1 |n\rangle \otimes I_2 |m\rangle = \hat{X}_1 |n\rangle \otimes |m\rangle$$

$$\underbrace{\hat{P}_2^{H_1^N \otimes H_2^M}}_{\substack{\text{Representation of } \hat{P}_2 \in H_2^M \\ \text{on } H_1^N \otimes H_2^M}} \rightarrow \quad (I_1 \otimes \hat{P}_2) | n \otimes m \rangle = I_1 |n\rangle \otimes \hat{P}_2 |m\rangle = |n\rangle \otimes \hat{P}_2 |m\rangle$$

In general if $\hat{\Omega}_1 \in H_1^N$; $\hat{\Lambda}_2 \in H_2^N$ operators; we have:

$$(\hat{\Omega}_1 \hat{\Lambda}_2)^{H_1^N \otimes H_2^M} \rightarrow (\hat{\Omega}_1 \otimes \hat{\Lambda}_2) | n \otimes n \rangle = \hat{\Omega}_1 |n\rangle \otimes \hat{\Lambda}_2 |m\rangle$$

We can also write it in a more compact way:

$$(\hat{X}_1 \otimes I_2) | n \otimes m \rangle = \hat{X}_1 | n \otimes m \rangle = \hat{X}_1 | nm \rangle$$

$$(\hat{\Omega}_1 \otimes \hat{\Lambda}_2) | n \otimes m \rangle = \hat{\Omega}_1 \hat{\Lambda}_2 | n \otimes m \rangle = \hat{\Omega}_1 \hat{\Lambda}_2 | nm \rangle$$

Suppose we are now into $|x\rangle$ basis ; $|x_1\rangle \otimes |x_2\rangle = |x_1 x_2\rangle$ if we apply $\hat{X}_1$ or $\hat{X}_2$ it will be resulted on eigenvalue. Thus if $\hat{X}_1$ is eigenvector of $|x_1\rangle$; $(\hat{X}_1 \otimes I_2)$ will be eigenvector of $|x_1\rangle \otimes |x_2\rangle$. In other words:

$$\hat{X}_1 |x_1 x_2\rangle = x_1 |x_1 x_2\rangle \quad \text{and}$$

$$\hat{X}_2 |x_1 x_2\rangle = x_2 |x_1 x_2\rangle$$

Also direct products of operators shares the following properties:

(1) $[\Omega_1 \otimes I_2 , I_1 \otimes \Lambda_2] = 0$ for any Ω_1, Λ_2 (operators of <u>each</u> particle comute)

(2) $(\Omega_1 \otimes \Gamma_2)(\theta_1 \otimes \Lambda_2) = (\Omega\theta)_1 \otimes (\Gamma\Lambda)_2$

(3) if $\underset{\in H_1^N}{[\Omega_1, \Lambda_1]} = \Gamma_1 \Rightarrow \underset{\in H_1^N \otimes H_2^M}{[\Omega_1^{H_1^N \otimes H_2^M}, \Lambda_1^{H_1^N \otimes H_2^M}]} = \Gamma_1 \otimes I_2$

(4) $\left(\Omega_1^{H_1^N \otimes H_2^M} + \Lambda_2^{H_1^N \otimes H_2^M}\right)^2 = (\Omega_1)^2 \otimes I_2 + I_1 \otimes (\Lambda_2)^2 + 2\,\Omega_1 \otimes \Lambda_2$

If we have some geral state vector $|\psi\rangle$ which consist of eigenvectors of some operator Ω in both spaces i.e:

$$|\psi\rangle = \sum_{w_1}\sum_{w_2} C_{w_1} C_{w_2} |w_1\rangle \otimes |w_2\rangle$$

If we project this into coordinate basis $|x_1\rangle \otimes |x_2\rangle$ we will have the following

$$\langle x_1 x_2 | \psi\rangle = \sum_{w_1}\sum_{w_2} C_{w_1} C_{w_2} \left(\langle x_1| \otimes \langle x_2|\right) \otimes \left(|w_1\rangle \otimes |w_2\rangle\right)$$

$$\underbrace{\langle x_1 x_2 | \psi\rangle}_{\text{wave-function}} = \sum_{w_1}\sum_{w_2} C_{w_1} C_{w_2} \langle x_1 | w_1\rangle \otimes \langle x_2 | w_2\rangle$$

eigen ~~value~~ functions in $|x\rangle$ basis just as usual.

$$\psi(x_1, x_2) = \sum_{w_1}\sum_{w_2} C_{w_1} C_{w_2}\, w_1(x_1)\, w_2(x_2)$$

Evolution of two-Particle state:

For two particles our hamiltonian of the system will assume the form:

$$\hat{H} = \frac{\hat{P}_1^2}{2m_1} + \frac{\hat{P}_2^2}{2m_2} + \hat{V}(\hat{X}_1, \hat{X}_2)$$ leading to two different classes:

1- H is separable ; $\hat{V}(\hat{X}_1,\hat{X}_2) = \hat{V}_1(\hat{X}_1) + \hat{V}_2(\hat{X}_2)$

$$\Rightarrow \hat{H} = \frac{\hat{P}_1^2}{2m_1} + \hat{V}_1(\hat{X}_1) + \frac{\hat{P}_2^2}{2m_2} + \hat{V}_2(\hat{X}_2) = \hat{H}_1 + \hat{H}_2$$

↳ particles interacts with external potential but they don't interact with each other. Thus their dynamics evolves idenpendently of each other which leed To idenpendently energy conservation.

For a stationary state $|\psi(t)\rangle = |E\rangle e^{-iEt/\hbar}$ where $|E\rangle = |E_1\rangle \otimes |E_2\rangle$

we find eigenvectors/values of $\hat{H}_1$ and $\hat{H}_2$ and them simply:

$$|\psi(t)\rangle = |E_1\rangle e^{-iE_1t/\hbar} \otimes |E_2\rangle e^{-iE_2t/\hbar}$$

If we would solve for coordinate basis its best to use separation of variables.