Functional Derivatives

João Lucas Rodrigues

March 21, 2022

1 Functional Derivative

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $f(x)$ be a function and F a functional $F[f]$. The functional derivative $\frac{\delta F}{\delta f(x)}$ at point x its defined to be the following:

$$\left(\frac{d}{d\varepsilon}F[f + \varepsilon\sigma]\right)_{\varepsilon=0} \equiv \int_{\mathbb{R}^n} d^n\sigma(x)\frac{\delta F}{\delta f(x)} \tag{1.1}$$

Where $\sigma(x)$ is any function and ε is a parameter.

The functional derivative have the following properties:

- $$\frac{\delta}{\delta f(x)}(c_1F_1 + c_2F_2) = c_1\frac{\delta F_1}{\delta f(x)} + c_2\frac{\delta F_2}{\delta f(x)}$$
- $$\frac{\delta}{\delta f(x)}(F_1F_2) = \frac{\delta F_1}{\delta f(x)}F_2 + F_1\frac{\delta F_2}{\delta f(x)}$$
- If $\Psi = \Psi(F)$ is a differential function of the functional F then we have the chain rule:
$$\frac{\delta\Psi}{\delta f(x)} = \frac{d\Psi}{dF}\frac{\delta F}{\delta f(x)}$$

For example lets consider

$$F[f] = \int_{\mathbb{R}^n} d^nxf(x)$$

So we have the following:

$$F[f + \varepsilon\sigma] = F[f] + \varepsilon F[\sigma] = F[f] + \varepsilon\int_{R^n} d^nx\sigma(x)$$

$$\implies \frac{d}{d\varepsilon}(F[f + \varepsilon\sigma])_{\varepsilon=0} = \int_{R^n} d^nx\sigma(x)$$

$$\implies \frac{\delta F}{\delta f(x)} = 1$$

Another example is to consider the following functional:

$$F[f] = \int d^n x d^n y K(x, y) f(x) f(y)$$

Until first order we have:

$$F[f + \varepsilon\sigma] = F[f] + \varepsilon \int d^n x d^n y K(x, y)\sigma(x) f(y) + \varepsilon \int d^n x d^n y K(x, y) f(x)\sigma(y)$$

$$\implies \left(\frac{d}{d\varepsilon} F[f + \varepsilon\sigma]\right)_{\varepsilon=0} = \int d^n x d^n y K(x, y)\sigma(x) f(y) + \int d^n x d^n y K(x, y) f(x)\sigma(y)$$

$$\implies \frac{\delta F}{\delta f(x)} = \int d^n y [K(x, y) + K(y, x)] f(y)$$

We also have functionals with parameters dependency, such as:

$$F_x[f] = \int_{\mathbb{R}^n} d^n x' K(x, x') f(x')$$

$$\frac{\delta F_x}{\delta f(y} = K(x, y)$$

With this we can construct:

$$f(x) = \int_{\mathbb{R}^n} d^n x' \delta(x - x') f(x')$$

$$\implies \frac{\delta f(x)}{\delta f(y)} = \delta(x - y)$$

$F[f_1, \ldots, f_n]$ n variables functional, the functional derivative $\frac{\delta F}{\delta f_k(x)}$ its build naturally by:

$$(\frac{d}{d\varepsilon} F[f_1 + \varepsilon\sigma_1, \ldots, f_N + \varepsilon\sigma_N])_{\varepsilon=0} = \int d^n x \sum_{k=1}^{N} \sigma_k(x) \frac{\delta F}{\delta f_k(x)}$$

And thus we can write the differential of the functional as:

$$\delta F = \int d^n x \sum_{k=1}^{N} \frac{\delta F}{\delta f_k(x)} \delta f_k(x)$$

2 Field Theory in Hamiltonian Formalism

Firstly using the differential we just saw we can write down the action variation (which is null) given a Lagrangian density $\mathcal{L}$. Here $x = x^\mu$

$$\frac{\delta S}{\delta \phi_\alpha(x)} = \frac{\partial \mathcal{L}}{\partial \phi_\alpha(x)} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha(x)}\right) - \nabla\left(\frac{\partial \mathcal{L}}{\partial(\nabla \phi_\alpha(x))}\right)$$

In which we can write down a compact form of Lagrange equations as

$$\frac{\delta S}{\delta \phi_\alpha(x)} = 0$$

The canonical momentum conjugate to the field $\phi_\alpha(x)$ is defined as:

$$\pi^\alpha(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha(x)}$$

And we define Hamiltonian density $\mathcal{H}$ by:

$$\mathcal{H} = \sum_\alpha \pi^\alpha \dot{\phi}_\alpha - \mathcal{L}$$

And the hamiltonian of the system is given by:

$$H[\phi_\alpha, \pi^\alpha] = \int d^3x \mathcal{H}(\oint_f(), \fint \oint_f, \bar{\jmath}^f(), \fint \bar{\jmath}^f())$$

The action is written as:

$$S = \int_\Omega d^4x \{ \sum \pi^\alpha \dot{\phi}_\alpha - \mathcal{H} \}$$

$$\delta S = \int_\Omega d^4x \sum_\alpha \left\{ \pi^\alpha \delta \dot{\phi}_\alpha + \delta \pi^\alpha \dot{\phi}_\alpha - \frac{\partial \mathcal{H}}{\partial \phi_\alpha} \delta \phi_\alpha - \frac{\partial \mathcal{H}}{\partial \nabla \phi_\alpha} \delta(\nabla \phi_\alpha) - \frac{\partial \mathcal{H}}{\partial \pi^\alpha} \delta \pi^\alpha - \frac{\partial \mathcal{H}}{\partial (\nabla \pi^\alpha) \delta (\nabla \pi^\alpha)} \right\}$$

$$= \int_\Omega d^4x \sum_\alpha \left\{ \left(-\dot{\pi}^\alpha - \frac{\partial \mathcal{H}}{\partial \phi_\alpha} + \nabla \frac{\partial \mathcal{H}}{\partial (\nabla \phi_\alpha)} \right) \delta \phi_\alpha + \left(\dot{\phi}_\alpha - \frac{\partial \mathcal{H}}{\partial \pi^\alpha} + \nabla \frac{\partial \mathcal{H}}{\partial (\nabla \pi^\alpha)} \right) \delta \pi^\alpha \right\}$$

Which leed us to Hamilton Equations by letting $\delta S = 0$

$$\dot{\phi}_\alpha = \frac{\partial \mathcal{H}}{\partial \pi^\alpha} - \nabla \frac{\partial \mathcal{H}}{\partial (\nabla \pi^\alpha)}; \tag{2.1}$$

$$\dot{\pi}^\alpha = -\frac{\partial \mathcal{H}}{\partial \phi_\alpha} + \nabla \frac{\partial \mathcal{H}}{\partial (\nabla \phi_\alpha)} \tag{2.2}$$

Or in a condensed form:

$$\dot{\phi}_\alpha(x) = \frac{\delta \mathcal{H}}{\delta \pi^\alpha(x)}; \tag{2.3}$$

$$\dot{\pi}^\alpha = -\frac{\delta \mathcal{H}}{\delta \phi_\alpha(x)} \tag{2.4}$$