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Einstein's Equations

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Abstract: Reference to chapter 4 of General Relativity by Robert M. Wald

Contents

A Motivation

We begin by first motivating the need for Einstein's Equations, and the passage from special relativity to general relativity. It's settled since SR how particles move thoughout spacetime and the laws in which they obey, however we never took gravity into the account. Why? Because gravity is not a force or a force field as imagined first by Newtonian mechanics, so we can't acctualy consider it's action though a field but rather spacetime itself which will leed us later on to a curve spacetime.

Let's first start with the ideia of Equivalence Principle. This state not only that the inertial mass is the same as the gravitational mass but also that there is no diffenrece between the behaviour of a free faling object (free particle) in a inertial frame or a accelerated frame (by gravity).

If we have an observer inside a box on a free fall to earths surface not only he will most likely die but also on his frame of reference, he is unable to tell if he is in a non inertial frame or not (wihtou an acelerometer). Alas, inside this elevator if we watches the free fall of a particle the particle behaves like we would expected it would in a inertial frame. .

revise and comple- ment this discussion

But why this is so important? The reason lies in Differential Geoemtry. By definition a free particle have null 4- acceleration and thus :

$$a^c = D_u u^c = 0 \tag{A.1}$$

But this is just the paralell transport of the tangent vector u^c along itself, which leed us to the geodesic equation of motion for a free particle

$$u^a \nabla_a u^c = 0 \tag{A.2}$$

We will have the in both reference frames, the free particles will follow two different geodesics, if we draw them we will see that they depart from each other with a rate of change different from zero, meaning that this departing have an acceleration. From different geometry this is related to Riemann tensor R^d_{abc} which have all the information on the curvature of our space. Also we have that the Riemann tensor can be written in terms of our derivative operator (thinking about the one stablished by the metric) and so our curvature and thus the gravity can be described thoughout the metric tensor.

insert geodesics

To better see this lets calculate the geodesic deviation equation: suppose we have a family of geodesics parametirezed by the parameter t and we define $T^a = (\frac{\partial}{\partial t})^a$ to be the tangent vector field of some geodesic, obviusly it satisfies the geodesic equation

$$T^a \nabla_a T^b = 0 \tag{A.3}$$

We can also define some other vector field to be some infinitesimal displacement between differents geodesics; $X^a = (\frac{\partial}{\partial s})^a$. Since both T^a and X^a are coordinate vector fields, they commute:

$$T^b \nabla_b X^a = X^b \nabla_b T^a \tag{A.4}$$

Which can be interpreted as the derivative of the displacement vector field along the geodesics, in other words is the notion of a "relative velocity" between geodesics, redefining that as:

$$v^a = T^b \nabla_b X^a \tag{A.5}$$

If we apply one more time our derivative operator we would be calculating the rate of change of this relative velocity between the geodesics, doing so we would have the following:

$$\begin{aligned}
a^a &= D_T v^a \\
&= T_c \nabla_c v^a \\
&= T_c \nabla_c \left(T^b \nabla_b X^a \right) \text{, using } \textbf{??} \\
&= T_c \nabla_c \left(X^b \nabla_b T^a \right) \\
&= \left(T^c \nabla_c X^b \right) (\nabla_b T^a) + X^b T^c \nabla_c \nabla_b T^a \text{, by addition and subtraction} \\
&= \left(X^c \nabla_c T^b \right) (\nabla_b T^a) + X^b T^c \nabla_c \nabla_b T^a + X^b T^c (\nabla_c \nabla_b - \nabla_b \nabla_c) T^a \\
&= \left(X^c \nabla_c T^b \right) (\nabla_b T^a) + X^b T^c \nabla_b \nabla_c T^a - R_{cbd}{}^a X^b T^c T^d
\end{aligned} \tag{A.6}$$

Note the following:

$$\left(X^c \nabla_c T^b \right) (\nabla_b T^a) + X^b T^c \nabla_b \nabla_c T^a = X^c \nabla_c \left(T^b \nabla_b T^a \right) \tag{A.7}$$

Which leed us to:

$$a^a = X^c \nabla_c \left(T^b \nabla_b T^a \right) - R_{cbd}{}^a X^b T^c T^d \tag{A.8}$$

Hoever by **??** we have finally:

$$a^a = -R_{cbd}{}^a X^b T^c T^d \tag{A.9}$$

And thus, the deviation of geodesics is determined by the curvature of space.
We must now assume that spacetime structure is a manifold M, with Lorentzian metric g_{ab}. Its possible to impose other two properties to our structure.

- Principle of General Covariance: the spacetime quantities which can be presented in laws of physics are only the metric $g_a b$ and objects directly derived from it.

- In the case where the metric is flat, zero curvature, the equations must be reduced to their cases in Special Relativity.

In order to find canditates to field equations in general relativity we must think about not only that they must simplify to their special relativity counterpart but also must satisfy newtonian gravity. We start by analising the energy stress momentum tensor T_{ab} in SR, for a perfect fluid we have the following expression:

$$T_{ab} = \rho u_a u_b + P\left(g_{ab} + u_a u_b\right) \tag{A.10}$$

We could think of a minimal substituition rule in order to pass form SR to GR in which we make the changes $\eta_{ab} \rightarrow g_{ab}$; $\partial_a \rightarrow \nabla_a$, however this is not a general rule; with these our perfect fluid satisfies the condition:

$$\nabla^a T_{ab} = 0 \tag{A.11}$$

Where P is pressure, ρ is density and u^a its 4-velocity. However this is true for all stress energy tensor fields, not only by a perfect fluid. For a Klein gordon scalar field:

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \left(\nabla_c \phi \nabla^c \phi + m^2 \phi^2\right) \tag{A.12}$$

And we might write Maxwell's equations in curved spacetime as:

$$\begin{cases} \nabla^a F_{ab} = -4\pi j_b \\ \nabla_{[a} F_{bc]} = 0 \end{cases} \tag{A.13}$$

Which yield the eletromagnetic stress tensor:

$$T_{ab} = \frac{1}{4\pi} \left\{ F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{de} F^{de} \right\} \tag{A.14}$$

If now we think about Newtonian gravitational field, its satisfies:

$$\nabla^2 \phi = 4\pi\rho \tag{A.15}$$

Where in ρ is giving informations on properties of the body genereting this gravitational field, in SR these informations are self contained in the stress energy momentum tensor, so we could incorporate these in GR by doing the analogy:

$$T_{ab} v^a v^b \leftrightarrow \rho \tag{A.16}$$

Also in newtonian gravity we have the effect of tidal force, which its acceleration can be described a $-(\vec{x}\cdot\vec{\nabla})\vec{\nabla}\phi$ where $\vec{x}$ is called the separation vector. Note however we have met something familiar to this prior, we calculated de acceleration of geodesics close do one another, if we change the vector fields T^a and X^a to a particles 4-velocity and displacement along a geodesic we would fall in a tidal acceleration in the case of general relativity, which we already know is governed by the curvature in the form of the riemann tensor. So we could incorporate in GR as:

$$R_{cbd}{}^{a}v^{c}v^{d} \leftrightarrow \partial_b\partial^a\phi \tag{A.17}$$

These two conditions suggest the following field equation

$$R_{ab} = 4\pi T_{ab} \tag{A.18}$$

But this equation does not hold. We know that we must have $nabla^a T_{ab} = 0$ and Bianchi's identity for the riemann tensor , if we use these two properties we find out that $\nabla_d R = 0$ which imples the stress tensor is constant thoughout the universe, which is obviously false. This equation was first proposed by Einstein in 1915, nevertheless in his paper 1915b he find out a way of correcting things to work with properties of stress tensor and Bianchy's identity if we consider the following field equation:

$$G_{ab} = 8\pi T_{ab} \tag{A.19}$$

Where

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \tag{A.20}$$

Known as Einstein tensor. These are in fact, Einsteins equations. With these correction now the Bianchi identity leed to local energy conservation as its expected from the stress tensor. Similar as the Maxwell Equations, the stress tensor T_{ab} can be interpreted as the source of the gravitational field. Not only that but the motion of a body can be found via $\nabla^a T_{ab} = 0$ and this condition (Fock 1939, Geroch and Jang 1975) implies that sufficiently "small" bodies whose self gravity is sufficiently "weak" must travel on a geodesic path. And also those body "large" enough to feel those tidal forces of gravitacional field will have their motion deviated from a geodesic.
To quote Wald:
" Spacetime is a manifold M on which there is defined a Lorentz metric g_{ab}. The curvature of g_{ab} is related to the matter distribuition in spacetime by Einstein's equation."

Acknowledgments

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Note added. This is also a good position for notes added after the paper has been written.

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